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Research Article

The Performance of Algorithm for Solving Constrained Optimization Problems

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Many engineering problems can be transformed into constrained optimization problems by establishing corresponding mathematical models. However, different solution algorithms for a constraint optimization problem usually show a significant disparity in the performance. Therefore, this paper proposes a hybrid genetic algorithm for solving constrained optimization problems. By constructing a special penalty function and combining with the random direction method, the selection, crossover and mutation operators are improved, thus enhancing the performance of the algorithm. In addition, this paper theoretically verifies the global convergence of the algorithm while employing elitism strategy.

Key words: Convergence of the algorithm, Significant disparity in the Performance.

INTRODUCTION

Optimization problems are frequently encountered in practical engineering and scientific experiments such as planning, prediction, controlling, diagnosis and process design. These problems are always under various constraint conditions and also have many solutions including traditional and modern ones. For the traditional solutions, there are analytic, direct and numerical methods that are complicated in calculation and require unimodality, derivability, continuity, *etc*. As to the modern ones, artificial neural network, simulated annealing, taboo search and intelligent evolutionary methods are included, which are intuitive, simple, operable, fast and efficient, in spite of some drawbacks [1-3]. In the middle of 1960s, based on the work of A.S. Fraser and H.J. Bremermann, John Holland from University of Michigan, the United States, proposed the bit-string encoding that is suitable for both crossover and mutation operations but mainly employs crossover as the genetic operator. In 1975, he published a pioneering work *Adaptation in Natural and Artificial Systems*. Afterwards, he and his students popularized and applied this algorithm in the optimization, machine learning, *etc.*, and named the algorithm as genetic algorithm officially [4-7]. In University of Michigan, a laboratory for genetic algorithms was established. Besides, the international society of genetic algorithms was founded. It has convoked International Conference on Genetic Algorithms biennially since 1985 and several conference papers have been published from then on. Obviously, the general coding technique and simple and effective operation of the genetic algorithm have laid a solid foundation for its success and extensive application. Based on these achievements, numerous researchers have improved this algorithm into a general model with adaptive process, which has been widely adopted in neural network, pattern recognition, image processing, machine learning, function optimization, *etc*.

The constrained optimization problems and the corresponding modified genetic algorithms

The mathematical model for constrained optimization problems

All constrained optimization problems can be transformed into the following mathematical model.

$$\begin{aligned} &\mathrm{Min} f(x) \\ &s.t.\ g_i(x) \le 0,\ i = 1,2,3,\cdots,p \\ &h_j(x) = 0,\ j = p+1, p+2,\cdots,m \qquad (1) \\ &x = (x_1, x_2, \cdots, x_n),\ a_i \le x_i \le b_i \end{aligned}$$

Where $f(x)$ is the function of $S(S=\prod_{i=1}^{n}[a_i,b_i]\subseteq R^n)$ to R, that is, the objective function of constrained optimization problems that are supposed to satisfy $g_i(x) \le 0$ and $h_j(x) = 0$.

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The processing of the constraint conditions of constrained optimization problems

For constrained optimization problems, they are generally solved by penalty function, gradient projection and multiplier methods. Myriads of simulation experiments reveal that the genetic algorithm (especially the modified one) exhibits high efficiency and great adaptability for this kind of problems. Generally, the penalty function method is employed to process the constraint conditions while solving constrained optimization problems utilizing the genetic algorithm. This method is also adopted to process the constraint conditions of the algorithm described in this paper.

The modified hybrid genetic algorithm with a newly constructed penalty function

Firstly, the following penalty function is constructed.

$$f(x)=\sum_{i=1}^{p}M_i\,|g_i(x)|+\sum_{j=p+1}^{m}N_j\max\{0,-h_j(x)\},$$

$$M_i=\frac{|g_i(x)|}{\sum_{i=1}^{p}|g_i(x)|+\sum_{j=p+1}^{m}\max\{0,-h_j(x)\}}$$

$$N_j=\frac{\max\{0,-h_j(x)\}}{\sum_{j=p+1}^{m}\max\{0,-h_j(x)\}}$$

The real number encoding method is used in the modified genetic algorithm and the initial population is generated applying the short interval method. The concrete algorithm steps are as follows:

$L=0$;

for ($count=1$; $count <= Maxcount$; $count++$) $\{$

$P(0)=\text{initialize}\{x_1(0),\ x_2(0),\cdots,\ x_M(0)\}$;

$t=0$;

evaluate $f(x_1(t)),\ f(x_2(t)),\cdots,f(x_M(t))$;

$x_i(t)\in S$

$x_{best}(t)=\arg\min_{i\in[1,M]} f(x_i(t))$; $f_{best}=f(x_{best}(t))$;

$x_{worst}(t)=\arg\max_{i\in[1,M]} f(x_i(t))$;

$f_{worst}=f(x_{worst}(t))$;

while($|f_{best}-f_{worst}| >= \varepsilon$ $\{$

Hybrid: $x=\lambda\, x_{best}(t)+(1-\lambda)x_m(t)$;

Where $m\in[1,M]$ is a random number,

$\lambda=(\lambda_1,\lambda_2,\cdots\lambda_n)$,

and $\lambda_i\in[-0.5,1.5]$

if ($f(\mathbf{x}) < f_{worst}$) $\mathbf{x}_{worst}(t)=x$;

Derivation: $\mathbf{x}_m=\boldsymbol{\lambda}x_m(t)$;

$x_{best}(t)=\arg\min_{i\in[1,M]} f(x_i(t))$;

$f_{best}=f(x_{best}(t))$;

$x_{worst}(t)=\arg\max_{i\in[1,M]} f(x_i(t))$;

$f_{worst}=f(x_{worst}(t))$;

$t=t+1$;

$\}$

print $x_{best}(t)$; $L=L+1$;

$\}$

The convergence of the algorithm

There are lots of modified genetic algorithms for solving constrained optimization problems, and they have multiple merits comparing with the basic genetic algorithm. Nevertheless, most findings on the convergence of algorithms are illustrated by the results obtained through simulation experiments, while the convergence of modified genetic algorithms is barely explored. Therefore, this paper researches the convergence of the algorithm utilizing random variables and the corresponding limit theory and verifies that a global optimal solution can be achieved through the above modified algorithm.

Definition 1: Suppose that $\{\xi_n\}$ is a sequence of random variables in the probability space $\{\Omega, f, P\}$. If $\exists\ R.V.\xi$, for $\forall \varepsilon > 0$, $\lim_{n\to\infty} P\{\|\xi_n - \xi\| < \varepsilon\} = 1$. Then, it is said that the sequence $\{\xi_n\}$ converges to ξ in probability.

Definition 2: Assume that $\{D_n\}$ is a sequence of non-negative random variables in the probability space $\{\Omega, f, P\}$ and $P\{\lim_{n\to\infty} D_n = 0\} = 1$. For $\forall \varepsilon > 0$, $P\{\bigcap_{n=0}^{\infty}\bigcup_{m\ge n} D_m > \varepsilon\} = 0$. In this case, the sequence $\{D_n\}$ is considered to converge to 0 in probability 1.

Assumption 1: (1) The feasible region D of constraint problem (1) is a bounded closed set of R^n. (2) For any neighborhood of y, any point in D, its measure is a positive number. (3) For a constraint problem, its objective function $f(x)$ is continuous in the searching space $S \supseteq D$.

For $\forall \delta > 0$, let

$$N_1 = \{x \in D \,\|\, f(x) - f^*(x) | < \delta\},\ N_2 = D / N_1,$$

Where $f^*(x) = \min\{f(x), x \in D\}$. The population sequence $\{P(t)\}$ generated by the algorithm can be divided into two states. (1) If there is a or more points in $\{P(t)\}$ that belong to N_1, then, $\{P(t)\}$ is in the state of S_1. (2) If all the points in $\{P(t)\}$ belong to N_2, $\{P(t)\}$ is in state S_2.

Theorem 1: p_{ij} represents the probability of the population $P(t+1)$ in the state of S_i ($i = 1,2$) containing $P\{t\}$ is found to be in state S_j ($j = 1,2$) p_{ij}. Under assumption 1, for any population $P\{t\}$ in state S_1, $p_{11} = 1$ and for any population $P\{t\}$ in state S_2, $p_{22} = c, (0 < c < 1)$.

Proof: According to the selection steps of the algorithm, if $P(t) \in S_1$, then, $P(t+1) \in S_1$. That is, the population in state S_1 is not likely to evolve to that in state S_2, so $p_{11} = 1$.

According to assumption 1, $X^* = \{x \mid \arg\min_{x\in D} f(x)\} \ne \phi$ and $f(x)$ is continuous in D, so for any global optimal solution $x^* \in X^*$, $\exists r > 0, \ni$. For $\forall x \in D \cap \{x \mid \|x - x^*\| \le r\}$, $| f(x) - f(x^*) | < \frac{\delta}{2}$. Apparently, $\{x \in S : \|x - x^*\| \le r\} \cap D \subseteq N_1$.

When population $X(k)$ is in state S_2, ($\forall x' \in X(k)$), $\bar{x}$ is a descendant generated through the selection, crossover and mutation of the algorithm. Then, the probability $p_1 = d > 0, d > 0$ of $\bar{x} = x + \Delta\Theta$ is a very small constant, ($\Delta\Theta = (\Delta\vartheta_1, \cdots, \Delta\vartheta_n)^T \sim (U(a_1, b_1), \cdots, U(a_n, b_n))$.

Hence,

$$P\{\ \bar{x} \in \{x \in S : \|x - x^*\| \le r\} \cap D\ \}$$

$$= p_1\ P\{(x + \Delta\Theta) \in \{x \in S : \|x - x^*\| \le r \cap D\}\} \ge d \prod_{i=1}^{n} P\{| x_i + \Delta\Theta_i - x_i^* | \le r\} =$$

$$d\prod_{i=1}^{n}\int_{x_i^*-x_i-r}^{x_i^*-x_i+r}\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}dt \quad (2)$$

Where x_i^* and x_i are the i th components of x^* and x , respectively.

Then, $P^*(x)=P\{(x+\Delta\Theta)\in\{x\in S:\left\|x-x^*\right\|\le r\}\cap D\}$, $x\in D$.

Owing to $\{x\in S:\left\|x-x^*\right\|\le r\}\cap D$ is a non-empty bounded closed region and the Lebesgue measure is larger than 0, $P^*(x)>0$. According to equation (2), $P^*(x)<1$, so $0<P^*(x)<1$ when $x\in D$. In addition, $\Delta\Theta_i$ obeys normal distribution and is a continuous random variable, so $P^*(x)$ is continuous in D. Moreover, as D is a bounded closed region, there is $y\in D$ that makes $P^*(y)=\min\{P^*(x)\mid x\in D\}$, ($0<P^*(y)<1$) tenable.

Due to $\{x\in S:\left\|x-x^*\right\|\le r\}\cap D\subseteq N_1$,

$p_{21}=P\{(x+\Delta\Theta)\in N_1\}\ge P^*(x)=P\{(x+\Delta\Theta)\in\{x\in S:\left\|x-x^*\right\|\le r\}\cap D\}\ge P^*(y)$.
Let $1-P^*(y)=c$, and obviously, $0<c<1$. Owing to $p_{21}+p_{22}=1$, then $p_{22}=1-p_{21}\le 1-P^*(y)=c$.

Theorem 2: If $\{P(k)\}$ is a population sequence generated by the algorithm, there is at least one point in $P(0)$ which belongs to D. Under $x_k^*=\arg\min_{x\in P(k)\cap D}f(x)$ and assumption 1, the group sequence $\{X(k)\}$ converges to a global optimal solution of the constrained problem in probability 1.

Proof: $\forall\varepsilon>0$,

Let $p_k=P\{|f(x_k^*)-f(x^*)|\ge\varepsilon\}$, then, $p_k=\begin{cases}0,\exists j\in\{1,2,\cdots,k\},s.t.x_j^*\in N_1\\ \overline{p_k},x_j^*\in N_1,j=1,2,\cdots,k\end{cases}$. According to theorem 1, $\overline{p_k}=P\{x_k^*\notin N_1,j=1,2,\cdots,k\}=p_{22}^k\le c^k$, so $\sum_{k=1}^{\infty}p_k\le\sum_{k=1}^{\infty}c^k=\frac{c}{1-c}<\infty$. Based on Borel-Cantelli lemma [9], $P\{\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}\{\left\|f(x_k^*)-f(x^*)\right\|\ge\varepsilon\}\}=0$, which means that the population sequence $\{X(k)\}$ converges to a global optimal solution of the constrained problem (1) in probability 1.

Assumption 2: (1) In the searching space, $D\subseteq R^n$, $S=[a_i,b_i]^n$, $i=1,2,\cdots,n$.
(2) $X^*=\{x^*\in D\mid f(x^*)=\min_{x\in S}f(x)\}\ne\phi$.
(3) $\forall\varepsilon>0$. Let $M_\varepsilon=\{x\in D\,\|\,f(x)-f(x^*)|\le\varepsilon,x^*\in X^*\}$, the Lebesgue measure of M_ε is $m(M_\varepsilon)$ ($m(M_\varepsilon)>0$).
(4) $0<f_{\min}\le f(x)\le f_{\max}<+\infty$.

Theorem 3: When, if $p_m>0$ (mutation probability) in the algorithm, when $\forall\varepsilon>0$, the probability of any individual generated through the crossover of the t th generation belonging to $x^*\in M_\varepsilon$ is $P\{x'=m(x)\in M_\varepsilon\}\ge\delta(t)>0$. When $\sum_{t=0}^{\infty}\delta(t)$ diverges and the elitism strategy is applied in the algorithm, under the assumption 2, the algorithm converges to a global optimal solution in probability, and the convergence has no relation with the initial population.

Proof: The probability of any individual x in the population generated by the crossover of the t th generation being selected for the mutation is $p_m>0$, so that the individual mutates into x' in the probability of $P\{x'=m(x)\in M_\varepsilon\}\ge\delta(t)>0$. As a result, in the t th generation, the probability of an individual belonging to M_ε is $\ge p_m\delta(t)>0$, so that of no individual belonging to M_ε is $P_{not}^t\le 1-p_m\delta(t)$. If $P_{not}(t)$ denotes the probability that there is no individual of the former t generations belonging to M_ε , then, $P_{not}(t)=\prod_{i=1}^{t}P_{not}^t\le\prod_{i=1}^{t}(1-p_m\delta(t))$.

Let $D_t = f_t^* - f^*$, where $f^* = \min\{f(x) \mid x \in S)\}$ and $f_t^* = \min\{f(x) \mid x \in P(t)\}$.

Due to the application of elitism strategy, $\forall \varepsilon > 0, P\{D_t > \varepsilon\} = P$(no individual generated from the former t generations belongs to $M_\varepsilon) = P_{not}(t) \le \prod_{i=1}^{t}(1 - p_m\delta(t))$.

As when $0 < p_m\delta(t) < 1$, $\prod_{i=1}^{t}(1 - p_m\delta(t)) \Leftrightarrow \sum_{i=1}^{t}\ln(1 - p_m\delta(t))$. Then, when $t \to \infty$,

$$\prod_{i=1}^{t}(1 - p_m\delta(t)) \to 0 \Leftrightarrow \sum_{i=1}^{t}\ln(1 - p_m\delta(t)) \to -\infty \Leftrightarrow \sum_{i=1}^{t} - \ln(1 - p_m\delta(t)) \to +\infty \Leftrightarrow \frac{-\ln(1 - p_m\delta(t))}{p_m\delta(t)} \to 1 \Leftrightarrow \sum_{t=0}^{\infty} p_m\delta(t) \text{ diverges.}$$

Furthermore, the condition $\sum_{t=0}^{\infty}\delta(t)$ diverges so that $\sum_{t=0}^{\infty} p_m\delta(t)$ diverges as well.

Therefore, $\prod_{t=1}^{\infty}(1 - p_m\delta(t)) = 0$ and $\lim_{t\to\infty} P(D_t > \varepsilon) = 0$, that is, the algorithm converges to the global optimal solution in probability.

Theorem 4: Under the condition of theorem 3, only if $\delta(t) \ge a > 0$ (t=0, 1, 2,...., and $0 < a < 1$), then the algorithm converges to a global optimal solution and the convergence is not influenced by the initial population.

Proof: Because $\delta(t) \ge a > 0$, then,

$$P_{not}(t) = P\{D_t > \varepsilon\} \le \prod_{i=1}^{t}(1 - p_m\delta(t))$$

$$\le \prod_{i=1}^{t}(1 - p_m a)$$

$$\le (1 - p_m)^t,$$

While $\sum_{t=1}^{N} P\{D_t > \varepsilon\} \le \sum_{t=1}^{N}(1 - p_m a)^t$, so $\sum_{t=1}^{\infty}(1 - p_m a)^t$ converges.

According to the relevance theory of series of positive terms and definitions 1 and 2, the algorithm converges to a global optimal solution and the convergence has nothing with the initial population. Theorem 5: The hybrid genetic algorithm with elitism strategy is employed and it converges to a global optimal solution in probability 1. Besides, the convergence is not associated with the initial population.

Proof: For $\forall \delta > 0$, let $N_1 = \{x \in D \| f(x) - f^*(x) |< \delta\}$, where $f^*(x) = \min\{f(x), x \in D\}$.

The event of the tth generation falling in N_1 is recorded as $U(\Theta_t)$. Given that the optimum reservation strategy is adopted, the optimal sequence $\{f(x_{t_k})\}$ in the hybrid genetic algorithm is monotonic without rising.

Therefore, $U(\Theta_1) \subseteq U(\Theta_2) \subseteq \cdots \subseteq U(\Theta_n) \subseteq \cdots$. Besides, owing $\{f(x_{t_k})\}$ is bounded and monotonic bounded sequences of numbers definitely have limit values, $\mathop{lin}\limits_{t\to\infty} P\{U(\Theta_t)\} = P\{\bigcup_{t=1}^{\infty} U(\Theta_t)\}$ =1. That is, the algorithm converges to a global optimal solution in probability 1 and is independent on the initial population.

Conclusion

This paper provides general methods for constrained optimization problems employing the genetic algorithm in the beginning. Then, a frame structure of using the modified hybrid genetic algorithm for solving constrained optimization problems is proposed. Moreover, the limit theory of random variables is used to discuss the conditions for the convergence of the algorithm. It is validated that under the elitism strategy, the proposed algorithm converges to a global optimal solution in probability 1, which is independent with the initial population.

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